

Matrix Subspaces of L_1 *

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Abstract

If $E = \{e_i\}$ and $F = \{f_i\}$ are two 1-unconditional basic sequences in L_1 with E r -concave and F p -convex, for some $1 \leq r < p \leq 2$, then the space of matrices $\{a_{i,j}\}$ with norm $\|\{a_{i,j}\}\|_{E(F)} = \left\| \sum_k \left\| \sum_l a_{k,l} f_l \right\| e_k \right\|$ embeds into L_1 . This generalizes a recent result of Prochno and Schütt.

1 Introduction

Recall that a basis $E = \{e_i\}_{i=1}^N$ of a finite ($N < \infty$) or infinite ($N = \infty$) dimensional real or complex Banach space is said to be K -unconditional if $\|\sum_i a_i e_i\| \leq K \|\sum_i b_i e_i\|$ whenever $|a_i| = |b_i|$ for all i . Given a finite or infinite 1-unconditional basis, $E = \{e_i\}_{i=1}^N$, and a sequence of Banach spaces $\{X_i\}_{i=1}^N$ denote by $(\sum \oplus X_i)_E$ the space of sequences $x = (x_1, x_2, \dots)$, $x_i \in X_i$, for which the norm $\|x\| = \|\sum_i \|x_i\| e_i\|$ is finite.

If X has a 1-unconditional basis $F = \{f_j\}$ then $(\sum \oplus X)_E$ can be represented as a space of matrices $A = \{a_{i,j}\}$, denoted $E(F)$, with norm

$$\|A\|_{E(F)} = \left\| \sum_i \left\| \sum_j a_{i,j} f_j \right\| e_i \right\|.$$

In [PS], Prochno and Schütt gave a sufficient condition for bases E and F of two Orlicz sequence spaces which assure that $E(F)$ embeds into L_1 . Here we generalize this result by giving a sufficient condition on two unconditional

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bases E, F , which assure that $E(F)$ embeds into L_1 . As we shall see this condition is also “almost” necessary.

Recall that an unconditional basis $\{e_i\}$ is said to be p -convex (resp. r -concave) with constant K provided that for all n and all x_1, x_2, \dots, x_n in the span of $\{e_i\}$,

$$\left\| \sum_{i=1}^n (|x_i|^p)^{1/p} \right\| \leq K \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}$$

(resp.

$$\left(\sum_{i=1}^n \|x_i\|^r \right)^{1/r} \leq K \left\| \sum_{i=1}^n (|x_i|^r)^{1/r} \right\|).$$

Here, for $x = \sum x(j)e_j$ and a positive α , $|x|^\alpha = \sum |x(j)|^\alpha e_j$.

L_p will denote here $L_p([0, 1], \lambda)$, λ being the Lebesgue measure. As is known and quite easy to prove, any 1-unconditional basic sequence in L_p , $1 \leq p \leq 2$ (resp. $2 \leq p < \infty$), is p -convex (resp. p -concave) with constant depending only on p . It is also worthwhile to remind the reader that any K -unconditional basic sequence in L_p is equivalent, with a constant depending only on p and K to a 1-unconditional basic sequence in L_p . It is due to Maurey [Ma] (see also [Wo, III.H.10]), that for every $1 \leq r < p \leq 2$, the span of every p -convex 1-unconditional basic sequence in L_1 embeds into L_p and also embeds into L_r after change of density; i.e., there exists a probability measure μ on $[0, 1]$ so that this span is isomorphic (with constants depending on r, p and the p -convexity constant only) to a subspace of $L_r([0, 1], \mu)$ on which the $L_r(\mu)$ and the $L_1(\mu)$ norms are equivalent.

If M is an Orlicz function then the Orlicz space ℓ_M embeds into L_p if and only if $M(t)/t^p$ is equivalent to an increasing function and $M(t)/t^2$ is equivalent to a decreasing function. This happens if and only if the natural basis of ℓ_M is p -convex and 2-concave.

Theorem 1 below states in particular that if E and F are two 1-unconditional basic sequences in L_1 with E r -concave and F p -convex for some $1 \leq r < p \leq 2$ then $E(F)$ embeds into L_1 . When specializing to Orlicz spaces, this implies the main result of [PS].

2 The main result

Theorem 1 *Let $E = \{e_i\}$ be a 1-unconditional basic sequence in L_1 with $\{e_i\}$ r -concave with constant K_1 and let X be a subspace of $L_1([0, 1], \mu)$ for*

some probability measure μ satisfying $\|x\|_{L_r([0,1],\mu)} \leq K_2 \|x\|_{L_1([0,1],\mu)}$ for some constant K_2 and all $x \in X$. Then $(\sum \oplus X)_E$ embeds into L_1 with a constant depending on K_1, K_2 and r only.

Consequently, if $E = \{e_i\}$ and $F = \{f_i\}$ are two 1-unconditional basic sequences in L_1 with E r -concave with constant K_1 and F p -convex with constant K_2 , for some $1 \leq r < p \leq 2$, then the space of matrices $A = \{a_{k,l}\}$ with norm

$$\|A\|_{E(F)} = \left\| \sum_k \left\| \sum_l a_{k,l} f_l \right\| e_k \right\|$$

embeds into L_1 with a constant depending only on r, p, K_1 and K_2 .

Proof: The p -convexity of $\{f_i\}$ implies that after a change of density the L_1 and L_r norms are equivalent on the span of $\{f_i\}$. See [Ma]. That is, there is a probability measure μ on $[0, 1]$ and a constant K_3 , depending only on r, p and K_2 such that $\|\sum a_j \tilde{f}_j\|_{L_r([0,1],\mu)} \leq K_3 \|\sum a_j \tilde{f}_j\|_{L_1([0,1],\mu)}$ for some sequence $\{\tilde{f}_j\}$ 1-equivalent, in the relevant L_1 norm, to $\{f_j\}$, and for all coefficients $\{a_i\}$. It thus follows that the second part of the theorem follows from the first part.

To prove the first part, in $L_1([0, 1] \times [0, 1], \lambda \times \mu)$ consider the tensor product of the span of $\{e_i\}$ and X , that is the space of all functions of the form $\sum_i e_i \otimes x_i$, $x_i \in X$ for all i , where $e_i \otimes x_i(s, t) = e_i(s)x_i(t)$. Then, by the 1-unconditionality of $\{e_i\}$ and the triangle inequality,

$$\begin{aligned} \left\| \sum_i e_i \otimes x_i \right\|_1 &= \int \left\| \sum_i |x_i(t)| e_i \right\|_{L_1([0,1],\lambda)} d\mu(t) \\ &\geq \left\| \sum_i \left(\int |x_i(t)| d\mu(t) \right) e_i \right\|_{L_1([0,1],\lambda)} \\ &= \left\| \sum_i \|x_i\| e_i \right\|. \end{aligned}$$

On the other hand, by the 1-unconditionality and the r -concavity with con-

stant K_1 of $\{e_i\}$ (used in integral instead of summation form),

$$\begin{aligned}
\left\| \sum_i e_i \otimes x_i \right\|_1 &= \int \int \left| \sum_i |x_i(t)| e_i(s) \right| d\lambda(s) d\mu(t) \\
&\leq \left(\int \left(\int \left| \sum_i |x_i(t)| e_i(s) \right|^r d\lambda(s) \right) d\mu(t) \right)^{1/r} \\
&= \left(\int \left\| \sum_i |x_i(t)| e_i \right\|_{L_1([0,1],\lambda)}^r d\mu(t) \right)^{1/r} \\
&\leq K_1 \left\| \sum_i \left(\int |x_i(t)|^r d\mu(t) \right)^{1/r} e_i \right\|_{L_1([0,1],\lambda)} \\
&\leq K_1 K_2 \left\| \sum_i \int |x_i(t)| d\mu(t) e_i \right\|_{L_1([0,1],\lambda)} \\
&= K_1 K_2 \left\| \sum_i \|x_i\| e_i \right\|
\end{aligned}$$

■

As is explained in the introduction the main result of [PS] follows as corollary.

Corollary 1 *If M and N are Orlicz functions such that $M(t)/t^r$ is equivalent to a decreasing function, $N(t)/t^p$ is equivalent to an increasing function and $N(t)/t^2$ is equivalent to a decreasing function then $\ell_M(\ell_N)$ embeds into L_1 .*

Remark 1 *The role of L_1 in Theorem 1 can easily be replaced with L_s for any $1 \leq s \leq r$.*

Remark 2 *If the bases E and F are infinite, say, and the smallest r such that E is r -concave is larger than the largest p such that F is p -convex, then $E(F)$ doesn't embed into L_1 . This follows from the fact that in this case it is known that ℓ_r^n uniformly embed as blocks of E and ℓ_p^n uniformly embed as blocks of F , for some $r > p$, while it is known that in this case $\ell_r^n(\ell_p^n)$ do not uniformly embed into L_1 .*

This still leaves the case $r = p$, which is not covered in Theorem 1, open:

- *If E and F are two 1-unconditional basic sequences in L_1 with E r -concave and F r -convex, does $E(F)$ embed into L_1 ?*

In the case that E is an Orlicz space the problem above has a positive solution. We only sketch it. By the factorization theorem of Maurey mentioned above ([Wo, III.H.10] is a good place to read it), and a simple compactness argument (to pass from the finite to the infinite case), it is enough to consider the case that F is the ℓ_r unit vector basis. If the basis of ℓ_M is r -concave, then the $2/r$ -convexification of ℓ_M (which is the space with norm $\|\{|a_i|^{2/r}\}\|_{\ell_M}^{r/2}$) embeds into $L_{2/r}$. This is again an Orlicz space, say, $\ell_{\tilde{M}}$. Now, tensoring with the Rademacher sequence (or a standard Gaussian sequence) we get that $\ell_{\tilde{M}}(\ell_2)$ embeds into $L_{2/r}$. We now want to $2/r$ concavify back, staying in L_1 , so as to get that $\ell_M(\ell_r)$ embeds into L_1 . This is known to be possible (and is buried somewhere in [MS]): If $\{x_i\}$ is a 1-unconditional basic sequence in L_s , $1 < s \leq 2$ then its s -concavification (which is the space with norm $\|\{|a_i|^{1/s}\}\|_{\ell_M}^s$) embeds into L_1 . Indeed, Let $\{f_i\}$ be a sequence of independent $2/s$ symmetric stable random variables normalized in L_1 and consider the span of the sequence $\{f_i \otimes |x_i|^s\}$ in L_1 .

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